PACE Electrical Engineering

Boolean Algebra

Introduction

Boolean Algebra is a branch of mathematics that formalizes logical reasoning. It establishes a methodology for evaluating statements or expressions that can be either true or false. For example, consider the following statements:

P: Joe is at least 18 years old

Q: Joe is a US citizen

R: Joe can vote for US President

Each of the statements is either true of false. Joe is either at least 18 years old or he is not. Joe is a US citizen or he isn't. Joe can vote for President or he can't. Contrast that with a statement like: Please go to the store. This statement is neither true nor false.

Notice that there is a relationship between P and Q and R. Joe can vote for US President if Joe is at least 18 years old AND Joe is a US citizen. So, whether or not Joe can vote depends on his age and citizenship. There are several possibilities, which are illustrated in the table below. Each row represents one possibility for Joe's age and citizenship and the resulting truth value of his voting eligibility.

P: Joe is at least 18 years old	**Q: Joe is a US citizen**	**R: Joe can vote for US President**
False	False	False
False	True	False
True	False	False
True	True	True

The above table is referred to as a truth table. We can express the relationship between P and Q and R as

$$R = P \text{ AND } Q.$$

AND is a Boolean or logic operator. It is characterized by the truth table above. R is True only if both P and Q are True. Otherwise, R is False.

Another logic function is OR. It is illustrated by the relationship among the following three statements:

A: I have at least $230 in cash

B: I have a credit card

C: I can buy an iPod touch

I can buy an iPod touch if I have at least $230 in cash OR I have a credit card (or I have at least $230 in cash and a credit card). The OR operator means something a little different from what we mean in common English. If you were to say that Elaine can either have cake or pie, we usually mean that she has to choose between cake and pie, she can't have both. In Boolean Algebra it means that she can have just cake or she can have just pie, or she can have both. The truth table for the iPod touch purchase is below:

***A*: I have at least $230 in cash**	***B*: I have a credit card**	***C*: I can buy an iPod touch**
False	False	False
False	True	True
True	False	True
True	True	True

Mathematically, we write that

$$C = A \text{ OR } B.$$

A, B, C, P, Q, and R are called Boolean or logical variables. They are variables that can have the values True or False.

A third Boolean operator is NOT. It is characterized by the formula $Y = \text{NOT } X$. So, if X is True then Y is False and if X is False then Y is True. As an example, consider

X: I like rock music

Y: I don't like rock music

The truth table for the NOT operator is

***X*: I like rock music**	***Y*: I don't like rock music**
False	True
True	False

Notation

We use the following notation for Boolean operators. We use multiplication for AND and plus for OR.

A* AND *B	***A* OR *B***	**NOT *A***
AB or $A \bullet B$	$A + B$	$\bar{A}$

We do this because Boolean expressions usually combine in the same way as these mathematical expressions.

Consider this next example:

A: I have enough money for a movie

B: My mom can give me a ride

C: My dad can give me a ride

D: I can go to the movies

In this example, I can go to the movies if my mom OR my dad can give me a ride AND I have enough money for the movie. We can express this as follows:

$$D = (B + C)A$$

If we rearrange this expression using regular algebra, we get

$$D = BA + CA,$$

which is interpreted as

> I can go to the movies if my mom can give me a ride and I have enough money for a movie
> OR my dad can give me a ride and I have enough money for a movie.

We can see these expressions are equivalent. To prove that these two expressions for D are equivalent, we would create a truth table for each one. If the two truth tables are the same, then the expressions are equivalent. To illustrate this, let's see whether the following two expressions are equivalent

NOT (A OR B)

(NOT A) AND (NOT B)

Let's first rewrite them using the mathematical notation. That's the way we'll work with Boolean expressions from now on:

$$\overline{A + B}$$

$$\bar{A}\bar{B}$$

Let's do the truth table for the first expression:

A	B	$A + B$	$\overline{A + B}$
False	False	False	True
False	True	True	False
True	False	True	False
True	True	True	False

Note: The first two columns in this truth table are determined by listing every combination of values of A and B. The remaining columns are determined by the values in the first two rows. We compute the $A + B$ column as an intermediate step to make it easier to compute the last column.

and now for the second expression:

$\boldsymbol{A}$	$\boldsymbol{B}$	$\boldsymbol{\bar{A}}$	$\boldsymbol{\bar{B}}$	$\boldsymbol{\bar{A}\bar{B}}$
False	False	True	True	True
False	True	True	False	False
True	False	False	True	False
True	True	False	False	False

Note: The first two columns in this truth table are determined by listing every combination of values of A and B. The remaining columns are determined by the values in the first two rows.

So, if the truth tables for two expressions are the same, the two expressions are equivalent.

One final note: There are ways of proving Boolean expressions are equivalent or not without using truth tables. But, we won't cover that here.