# Functions

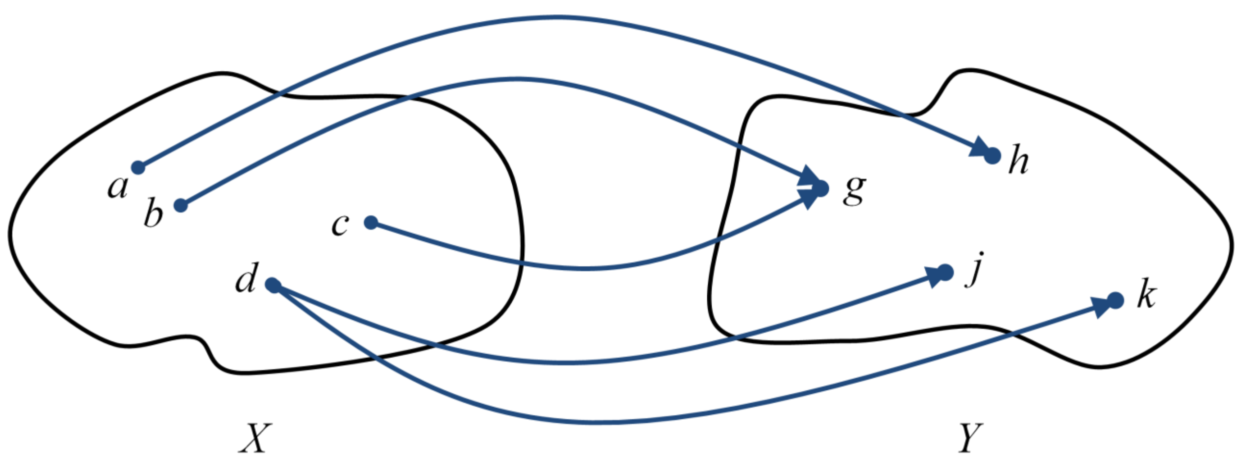
## Definitions

**Set** – A set is a collection of items

- Ways of specifying a set:
    - List the elements: {1, 2, 4, 8, 48}, {ball, cake, mouse, car}
    - Describe the elements: {All PACE students}, $\{x \mid x \geq 4\}$

  Listing all the elements is fine for small finite sets. Describing the elements is appropriate for large or infinite sets
- Sets can be finite or infinite

**Relation** – A relation is a mapping of elements from one set to an element or elements in another set




- A point in one set can map uniquely to a point in another set
- Two different points in one set can map to the same point in the other set
- One point can map to two different points (Ex. $y = x^{1/2}$)
- Relations can be expressed as a sequence of ordered pairs: $(a, h)$, $(b, g)$, $(c, g)$, $(d, j)$, $(d, k)$

**Function** – A function $f$ from set $X$ to set $Y$ is a relation such that for each element $y$ in $Y$ there is at most one $x$ in $X$ for which $f(x) = y$.

- The figure above does not represent a function, because there are two points in $X$, $b$ and $c$, which map into the same point in $Y$, $g$.

**Domain (of a function)** – The domain of a function on a set $X$ is the set of elements of $X$ for which the function is defined.

- The set of points for which a function is defined may be explicitly stated: $f(x) = 3x$ for $x \geq 5$ (so the domain is $\{x \mid x \geq 5\}$)
- If not explicitly stated then the domain must be inferred from the formula for the function. To determine where the function is defined, it is often easier to determine where it is undefined and take the complement of that set. To determine where is set is undefined look for the following:
  - where the denominator of fraction is equal $0$,
  - where the argument of a square root is negative. (There's an implicit assumption that we are dealing with functions of real variables.)
- Examples:
  - $f(x) = \frac{1}{x-1}$, This is undefined for $x = -1$. So the domain is the complement: $\{x \mid x \neq -1\}$
  - $f(x) = \sqrt{x-12}$, This is undefined for $x > 12$. So, the domain is $\{x \mid x \leq 12\}$
- The domain is also called the inverse image of a function

**Range (of a function)** – The range of a function is the set of values that the function can obtain.

- One way to construct the range of a function is to determine whether certain key values (or types of values) can be obtained. Can the function be zero? Can it be positive? Can it be negative? Is there a bound or does it continue toward positive or negative infinity?
- Example:
  - $f(x) = \frac{1}{x-1}$
    - Can the function be zero? Since the number is a constant, it can never be $0$ for any value of $x$. So, the function can never be $0$. Zero is not in the range.
    - Can the function be positive? The numerator is always positive and the denominator is positive for $x > 1$. So, the function is positive for $x > 1$.
    - Can the function be negative? The numerator is always positive and the denominator is negative for $x < 1$. So, the function is negative for $x < 1$.
    - Is the function bounded? As $x$ gets closer and closer to $1$ the denominator gets closer and closer to $0$, so the function gets larger and larger. Since the denominator is positive for $x$ near $1$, but greater than $1$, the function is unbounded on the positive side. Since the denominator is negative for $x$ near $1$, but less than $1$, the function is unbounded on the negative side.
    - So, the function can obtain any value except 0. The range is $\{y \mid y \neq 0\}$. Another way to write it is $\{-\infty < y < 0 \ \cup 0 < y < \infty\}$.
- The range is also called the image of the function